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# Comment on "Minimal parabolic quantum groups in twist deformations" by M. Ilyin and V. Lyakhovsky

# On a "new" deformation of GL(2)

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**Abstract.** We refute a recent claim in the literature [Czech. J. Phys. **56**, 1191 (2006)] of a "new" quantum deformation of GL(2).

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Until the year 2000 it was not clear how many distinct quantum group deformations are admissible for the group $GL(2)$ and the supergroup $GL(1|1)$. For the group $GL(2)$ there were the well-known standard $GL_{pq}(2)$ [1] and nonstandard (Jordanian) $GL_{gh}(2)$ [2] two-parameter deformations. (The dual quantum algebras of $GL_{pq}$ and $GL_{gh}$ were found in [3] and [4], respectively.) For the supergroup $GL(1|1)$ there were the standard $GL_{pq}(1|1)$ [5–7] and the hybrid (standard-nonstandard) $GL_{qh}(1|1)$ [8] two-parameter deformations.

Then, in the year 2000 in [9] it was shown that the list of these four deformations is exhaustive (refuting a long standing claim of [10] (supported also in [11,12]) for the existence of a hybrid (standard-nonstandard) two-parameter deformation of $GL(2)$). In particular, it was shown that the above four deformations match the distinct triangular $4\times 4$ $R$-matrices from the classification of [13] which are deformations of the trivial $R$-matrix (corresponding to undeformed $GL(2)$)[1].

At the end of the Introduction of [9] one can read the following:
"Instead of briefly stating the equivalence of the hybrid (standard-nonstandard) of [10] with the standard $GL_q(2)$, we have chosen to present our elementary analysis explicitly and in some detail. We consider this worthwhile for dissipating some confusions. Several authors have presented attractive looking hybrid deformations without noticing disguised equivalences. We ourselves devoted time and effort to their study before reducing them to usual deformations. We hope that our analysis will create a more acute awareness of traps in this domain".

In spite of this there still appear statements about "new" deformations of $GL(2)$. In particular, in the Conclusions of the paper [14] we read:
"Thus, we have a new quantization of $GL(2)$ that is neither a twist deformation nor a quasitriangular one."

Unfortunately, the authors of [14] have not noticed that their "new" quantization of $GL(2)$ is actually a partial case of the two-parameter nonstandard (Jordanian) $GL_{gh}(2)$ deformation [2].

It is easy to demonstrate this explicitly. First we repeat the relations for the four generators $a,b,c,d,$ of deformed $GL(2)$ from the paper [14]:
The co-product is standard:

$$\Delta(a) = a\otimes a + b\otimes c$$

$$\Delta(b) = a\otimes b + b\otimes d$$

$$\Delta(c) = c\otimes a + d\otimes c$$

$$\Delta(d) = c\otimes b + d\otimes d \tag{1}$$

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[1] Superficially, there are seven triangular $4\times 4$ $R$-matrices in [13], however, three of them are special cases of the essential four, cf. [9]

while the algebra relations given in (10) of [14] are:

$$[a,b]=b^2,\qquad [a,c]=0,$$
$$[b,c]=-db,\qquad [b,d]\ =\ 0,$$
$$[a,d]=db,\qquad [c,d]\ =\ d^2-ad+cb. \tag{2}$$

On the other hand the two-parameter nonstandard (Jordanian) $GL_{gh}(2)$ deformation [2] is given as follows. The co-product is the standard one given above in (1), while the algebra relations are $(g,h\in\mathbb{C})$:

$$[d,c]=hc^2,\qquad [d,b]\ =\ g(ad-bc+hac-d^2),$$
$$[b,c]=gdc+hac-ghc^2,\qquad [a,c]\ =\ gc^2,$$
$$[a,d]=gdc-hac,\quad [a,b]\ =\ h(da-bc+\ gdc-a^2). \tag{3}$$

It is easy to notice that (2) is a special case of (3) obtained for $g=0,\ h=1$.

To show this, as a first step, we set the latter values in (3) to obtain:

$$[d,c]=c^2,\qquad [d,b]\ =\ 0,$$
$$[b,c]=ac,\qquad [a,c]\ =\ 0,$$
$$[a,d]=-ac,\qquad [a,b]\ =\ da-bc-a^2. \tag{4}$$

Now we note that under the exchange:

$$a\longleftrightarrow d,\qquad b\longleftrightarrow c \tag{5}$$

the co-product (1) remains unchanged, while (4) becomes:

$$[a,b]=b^2,\qquad [a,c]\ =\ 0,$$
$$[c,b]=db,\qquad [d,b]\ =\ 0,$$
$$[d,a]=-db,\qquad [d,c]\ =\ ad-cb-d^2. \tag{6}$$

Clearly, (6) coincides with (2).

Thus, as anticipated there is no new deformation of $GL(2)$ in [14].

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